JOURNAL OF FUNCTIONAL ANALYSIS **82**, 138–150 (1989)

An Extremal Problem in Fourier Analysis with Applications to Operator Theory

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Communicated by D. Sarason

Received August 24, 1987

DEDICATED TO PROFESSOR BÉLA SZŐKEFALVI-NAGY ON HIS 75TH BIRTHDAY

A certain minimal extrapolation problem for Fourier transforms is known to have consequences for the determination of best possible bounds in some problems in linear operator equations and in perturbation of operators. In this paper we estimate the value of the constant in the Fourier-transform problem, by an analytic reformulation.

1. INTRODUCTION

In an earlier paper [4] it was shown that a certain minimal extrapolation problem in Fourier analysis can provide bounds on solutions of some linear operator equations, and that these in turn lead to some perturbation bounds for spectral subspaces of self-adjoint or normal operators. In this paper we supplement that discussion with further information on the value of these bounds.

* This author thanks the Indian Statistical Institute for a visiting appointment during which some of this work was done. All three authors acknowledge financial support from NSERC of Canada.

0022-1236/89 $3.00

In this introductory section we state the problems surveyed and the relations between them and outline the results to be presented.

Problem 1: *An Extremal Problem for the Fourier Transform on* $L_1(\mathbb{R})$

For $f \in L_1(\mathbb{R})$ denote its L_1 norm, as usual, by $\|f\|_1 = \int_{\mathbb{R}} |f(s)|\, ds$. Let $\hat{f}$ be the Fourier transform of f with the normalisation

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi s} f(s)\, ds.$$

Consider all those functions f in $L_1(\mathbb{R})$ for which $\hat{f}(\xi) = 1/\xi$ when $|\xi| \geqslant 1$. The problem is to find the infimum of $\|f\|_1$ over this class. In other words, we want to evaluate the number c_1 defined by

$$c_1 = \inf\left\{ \|f\|_1 : f \in L_1(\mathbb{R}), \hat{f}(\xi) = \frac{1}{\xi} \text{ whenever } |\xi| \geqslant 1 \right\}. \tag{1.1}$$

Problem 2: *An Extremal Problem for the Fourier Transform on* $L_1(\mathbb{R}^2)$

For $f \in L_1(\mathbb{R}^2)$ let $\hat{f}$ denote its Fourier transform with the normalisation

$$\hat{f}(\xi_1, \xi_2) = \iint_{\mathbb{R}^2} e^{-i\xi s} f(s_1, s_2)\, ds_1\, ds_2.$$

Here ξs stands for the real inner product $\xi_1 s_1 + \xi_2 s_2$. We will write s for the vector (s_1, s_2) in $\mathbb{R}^2$ as well as for the complex number $s_1 + is_2$, and let $|s| = (s_1^2 + s_2^2)^{1/2}$. A two variable analogue of Problem 1 is the problem of finding the number c_2 defined by

$$c_2 = \inf\left\{ \|f\|_1 : f \in L_1(\mathbb{R}^2), \hat{f}(\xi) = \frac{1}{\xi_1 + i\xi_2} \text{ whenever } |\xi| \geqslant 1 \right\}. \tag{1.2}$$

The next two problems concern the equation $AQ - QB = S$, where $A \in \mathscr{B}(\mathscr{H})$ the space of bounded operators on a Hilbert space $\mathscr{H}$, and $B \in \mathscr{B}(\mathscr{K})$, where $\mathscr{K}$ is another Hilbert space. It is well known that if the spectra $\sigma(A)$ and $\sigma(B)$ of A and B are disjoint subsets of the plane then for every $S \in \mathscr{B}(\mathscr{K}, \mathscr{H})$ the above equation has a unique solution Q in $\mathscr{B}(\mathscr{K}, \mathscr{H})$. In other words if $I_{A,B}$ denotes the operator (or "transformer") from $\mathscr{B}(\mathscr{K}, \mathscr{H})$ into itself defined as $I_{A,B}(Q) = AQ - QB$ then $I_{A,B}$ is invertible whenever $\sigma(A)$ and $\sigma(B)$ are disjoint. In [4] the authors obtained some information on bounds for the norm $\|(I_{A,B})^{-1}\|$, when A, B are self-adjoint or normal. The problems described below concern this question. We will use the notations $\|A\|$ for the usual operator norm and $|||A|||$ for any unitarily invariant norm on $\mathscr{B}(\mathscr{K}, \mathscr{H})$. (See [7].)

Problem 3: *The Operator Equation* $AQ - QB = S$ *with Self-Adjoint* A *and* B

Let K_A, K_B be two closed subsets of $\mathbb{R}$ such that $|s - t| \geqslant \delta$ for every $s \in K_A$ and $t \in K_B$, where δ is a positive number. Suppose A, B are self-adjoint and that $\sigma(A)$ and $\sigma(B)$ are contained in K_A and K_B, respectively. We know that $I_{A,B}$ is invertible. In [4] the authors obtained bounds of the type $\|(I_{A,B})^{-1}\| \leqslant c_1'/\delta$, where c_1' is a constant independent of A and B, and showed that the same estimate is valid for every unitarily invariant norm on $\mathscr{B}(\mathscr{H}, \mathscr{H})$. Put differently, the operator equation $AQ - QB = S$ has a unique solution Q for a given S under the above conditions. It is being asserted that for all operators as described and for all unitarily invariant norms $\delta\ |||Q||| \leqslant c_1'\ |||S|||$ for some constant c_1'. The problem is to find the *least* constant c_1' with this property. (In the subsequent discussion c_1' wil mean this smallest number.)

Problem 4: *The Operator Equation $AQ - QB = S$ with Normal A and B*

In the above discussion replace "self-adjoint" by "normal" and the real line by the complex plane. The problem is to find the least number c_2' for which we can generally assert $\|(I_{A,B})^{-1}\| \leqslant c_2'/\delta$; or in the other formulation, for which we always have $\delta\ |||Q||| \leqslant c_2'\ |||s|||$.

Problem 5: *Perturbation of Spectral Subspaces of Self-Adjoint Operators*

Let A, B be bounded self-adjoint operators on a Hilbert space $\mathscr{H}$. Let K_A, K_B be two closed subsets of $\mathbb{R}$ separated by a distance δ as in Problem 3. Let E be the spectral projector for A belonging to the set K_A and F the spectral projector for B belonging to the set K_B. We want to find the least positive number c_1'' such that for all operators as described and for all unitarily invariant norms we have $\delta\ |||EF||| \leqslant c_1''\ |||A - B|||$.

Problem 6: *Perturbation of Spectral Subspaces of Normal Opertors*

In Problem 5 replace "self-adjoint" by "normal" and $\mathbb{R}$ by $\mathbb{C}$. We seek the least positive number c_2'' for which the inequality $\delta\ |||EF||| \leqslant c_2''\ |||A - B|||$ always holds under the above conditions.

The results of the earlier study [4] included

$$c_1'' \leqslant c_1' \leqslant c_1 < 2,$$

$$c_2'' \leqslant c_2' \leqslant c_2 < \infty.$$

There is no substantial evidence for expecting that any of these inequalities is an equality. However, we do know that the constants on each line cannot differ too much.

Indeed, the simplest examples with $\dim \mathscr{H} = 2$ show that $1 \leqslant c_1''$. It is clear from the definitions that $c_1'' \leqslant c_2''$ and $c_1' \leqslant c_2'$. In [4] it was also shown that $c_1' > (3/2)^{1/2}$ and that $c_2'' \geqslant \pi/2$.

The constant c_2'' is related to a long-standing open problem in perturbation of eigenvalues [3], which we state below.

Problem 7: Variation of Eigenvalues of Normal Matrices

Let d be the least positive number with the property that if A, B are any two normal matrices with eigenvalues $\alpha_1, ..., \alpha_n$ and $\beta_1, ..., \beta_n$, respectively, then there exists a permutation σ of the indices such that

$$\max_i |\alpha_i - \beta_{\sigma(i)}| \leqslant d \|A - B\|.$$

The problem is to find d.

In [4] it was shown that $1 \leqslant d \leqslant c_2''$. In fact we have $d \leqslant c_2'''$ where c_2''' is the (possibly smaller) best constant which works in Problem 6 when limited only to the operator norm $\|\cdot\|$.

In particular $d \leqslant c_2$. To date one knows no other way of finding an upper bound for d.

There is a natural analogue of Problem 7 for the case of infinite-dimensional operators [2], and it is known [5] that the constant d which works in the finite-dimensional problem also works for this infinite-dimensional analogue.

A. McIntosh and A. Pryde [8] have used ideas similar to those in [4] to study commuting tuples of self-adjoint operators. Here the real analysis used in Problem 3 and the complex analysis used in Problem 4 are replaced by Clifford analysis. This then leads to an extremal problem in many-variable Fourier analysis analogous to the one- and two-variable problems mentioned above. This problem is not treated in this paper.

In Section 2 we will begin with the answer to Problem 1. This answer, unknown to us, was already in the literature long before we began this study. Section 2 also sets the stage for the general attack on problems of this type.

In Section 3 we will reduce Problem 2 to an equivalent problem in a single variable, somewhat resembling the one already solved.

In Section 4 we give an upper bound on the constant c_2. As explained above, this gives upper bounds for the constants occurring in the remaining problems as well.

2. Minimal Extrapolation

The study of the kind of extremal problems for the Fourier transform with which we are concerned was initiated in the 1930s by A. Beurling and B. Sz.-Nagy. Recent discussions may be found in [11, Chap. 7; 15] and the literature cited there.

The general context is the following. Let E be an open subset of $\mathbb{R}^n$ and let ϕ be a continuous function defined on its complement F. We seek an L_1 function f of minimal L_1 norm whose Fourier transform coincides with ϕ on the set F.

The minimality requirement means that $\|f+g\|_1 \geqslant \|f\|_1$ for all g whose Fourier transform is supported in E. This variational condition means that $\int \overline{(\operatorname{sgn} f)}\, g = 0$ and hence $\int \overline{(\operatorname{sgn} f)}{}^{\wedge}\, \hat{g} = 0$ for all such g. This in turn implies that $(\operatorname{sgn} f)^{\wedge}$ is a distribution whose support is contained in F. As Shapiro [11] points out, such reasoning does not necessarily provide a way to prove that a particular function is extremal, or even to find a likely candidate, but it may give some guidance.

Problem 1 stated in Section 1 was studied for different reasons, and solved, by Sz.-Nagy in 1938 in collaboration with A. Strausz. This paper not being widely available, Sz.-Nagy published a new exposition in 1953 [13]. We overlooked that paper and thank Professor Sz.-Nagy for bringing it to our attention. We will not reproduce his argument here, but quote his result: the infimum c_1 in (1.1) is $\pi/2$ and is attained by f such that $\operatorname{sgn} f(t) = \operatorname{sgn} \sin t$.

The paper by J. D. Vaaler [15] also discusses this problem and its applications to some questions in number theory.

3. A Reformulation of Problem 2

Before beginning our computations regarding c_2, let us comment on the simpler known fact that $c_2 < \infty$. The proof of this given in [4] was somewhat arduous. Later M. S. Narasimhan gave us the following especially transparent proof.

Let D be the disk $\{\xi\colon |\xi| \leqslant 1\}$ in the plane. It is required to exhibit an $f \in L_1(\mathbb{R}^2)$ for which $\hat{f}(\xi) = 1/(\xi_1 + i\xi_2)$ outside D. Let $\phi(\xi)$ be any C^∞ function which vanishes strongly at 0 (for example, which is zero in a neighbourhood of 0) and which is identically equal to 1 outside D. Define $\psi(\xi) = \phi(\xi)/\xi$. We will show that the inverse Fourier transform $\check{\psi}$ is in L_1. To this end, consider $\eta(\xi) = (d/d\bar{\xi})\, \psi(\xi) = (1/\xi)(d/d\bar{\xi})\, \phi(\xi)$. This is a C^∞ function with compact support and hence belongs to the Schwartz space $\mathscr{S}$. Hence $\check{\eta}$ also lies in $\mathscr{S}$. But $\check{\psi}(s) = 2i\check{\eta}(s)/s$ and hence belongs to L_1. This shows that $c_2 < \infty$.

Now for the computation of the value of c_2. Consider the tempered distribution $f_0(s) = -1/2\pi i s$. We know that $\hat{f}_0(\xi) = 1/\xi$. (See [10, p. 205].) We seek an element p of $\mathscr{S}'$ with the following properties:

(P1) $\hat{p}$ is an L_1 function with $\operatorname{Supp} \hat{p} \subset D$;

(P2) if we define the distribution f as

$$f = f_0 + p \tag{3.1}$$

then f is an L_1 function.

Note that $c_2 = \inf_p \|f\|_1$ over such p.

Writing $s = re^{i\theta}$ we see that

$$c_2 = \inf_p \frac{1}{2\pi} \int_0^\infty r\, dr \int_{-\pi}^{\pi} \left| \frac{1}{r} - ie^{i\theta} 2\pi p(s) \right| d\theta. \tag{3.2}$$

Our first aim is to reduce the number of variables in the problem by ascertaining the most favourable dependence of $p(s)$ on θ.

For each r separately, we have

$$\int_{-\pi}^{\pi} \left| \frac{1}{r} - ie^{i\theta} 2\pi p(s) \right| d\theta \geqslant \left| \frac{2\pi}{r} - 2\pi i \int_{-\pi}^{\pi} e^{i\theta} p(s)\, d\theta \right|,$$

with equality in the case (among others) that $e^{i\theta} p(re^{i\theta})$ is independent of θ. For any p let

$$F(r) = \int_{-\pi}^{\pi} ie^{i\theta} p(s)\, d\theta. \tag{3.3}$$

Then what has been shown is that

$$\int_{-\pi}^{\pi} \left| \frac{1}{r} - ie^{i\theta} 2\pi p(s) \right| d\theta \geqslant 2\pi \left| \frac{1}{r} - F(r) \right|,$$

with equality for functions p with the special angular dependence mentioned above. But any F obtainable in this way via (3.3)—say, from p_0—is also obtained from a p which has this special angular dependence, viz., from $p(s) = (1/2\pi i)\, e^{-i\theta} F(r)$. Further, this p satisfies properties (P1) and (P2) if p_0 does, for we can write $p = (1/2\pi) \int_{-\pi}^{\pi} e^{i\alpha} p_\alpha\, d\alpha$, where $p(re^{i\theta}) = p_0(re^{i(\theta+\alpha)})$. Consequently we can restrict our attention to those p which also satisfy this additional condition:

(P3) $sp(s)$ is a radial function.

We thus have

$$c_2 = \inf_F \int_0^\infty r\, dr \left| \frac{1}{r} - F(r) \right| = \inf_G \int_0^\infty |G(r)|\, dr, \tag{3.4}$$

where we have put

$$G(r) = 1 - rF(r); \tag{3.5}$$

the infimum in (3.4) being taken over all F defined via (3.3) with p satisfying conditions (P1), (P2), and (P3). The problem now has been cast as a one-variable L_1 minimisation problem. We now seek a useful characterisation of the class of functions G which enter.

By standard results in Fourier analysis [12] the condition (P3) implies that $\hat{p}$ also has a special angular dependence. If $\xi = \rho e^{i\phi}$ then we can write $\hat{p}$ in the form

$$\hat{p}(\xi) = -2\pi i e^{-i\phi} Q(\rho). \tag{3.6}$$

The restriction (P1) implies that $\operatorname{Supp} Q \subset [0, 1]$. Further since $\hat{p}(\xi) = \hat{f}(\xi) - \hat{f}_0(\xi) = \hat{f}(\xi) - 1/\xi$, and $\hat{f}$ is continuous being the Fourier transform of an L_1 function, we can write

$$Q(\rho) = \frac{1}{2\pi i\rho} + h(\rho), \tag{3.7}$$

where h is a continuous function.

We will now express F in terms of Q. We can obtain p from $\hat{p}$ by the Fourier inversion formula. Substituting this expression for p in (3.3) and noting that $s_1\xi_1 + s_2\xi_2 = r\rho\cos(\theta - \phi)$ we get

$$\begin{aligned}
F(r) &= \frac{i}{4\pi^2}\int_{-\pi}^{\pi} e^{i\theta}\,d\theta \int_{-\pi}^{\pi} d\phi \int_0^1 \exp(ir\rho\cos(\theta-\phi))\,\hat{p}(\xi)\rho\,d\rho \\
&= \int_{-\pi}^{\pi} e^{i\alpha}\,d\alpha \int_0^1 \exp(ir\rho\cos\alpha)\,Q(\rho)\,\rho\,d\rho \\
&= 2\int_0^{\pi} \cos\alpha\,d\alpha \int_0^1 \exp(ir\rho\cos\alpha)\,Q(\rho)\,\rho\,d\rho,
\end{aligned}$$

where the substitution $\alpha = \theta - \phi$ has enabled us to perform one of the integrations. Since $Q(\rho)$ has the form (3.7) the order of integration in the last double integral can be changed. Make this change and then substitute $\lambda = \rho\cos\alpha$ in the inner integral to get

$$\begin{aligned}
F(r) &= 4\int_0^1 Q(\rho)\,\rho\,d\rho \int_0^{\pi/2} i\sin(r\rho\cos\alpha)\cos\alpha\,d\alpha \\
&= 4i\int_0^1 Q(\rho)\,d\rho \int_0^{\rho} \sin r\lambda\,\frac{\lambda}{(\rho^2-\lambda^2)^{1/2}}\,d\lambda.
\end{aligned}$$

Now notice that

$$\int_0^1\int_0^{\rho} \frac{\lambda^2}{(\rho^2-\lambda^2)^{1/2}}\,|Q(\rho)|\,d\lambda\,d\rho < \infty,$$

because the inner integral is $\mathcal{O}(\rho^2)$ (actually const $\cdot\,\rho^2$) and $Q(\rho)$ is $\mathcal{O}(1/\rho)$ near 0. So, once again the order of integration in the last double integral can be changed, and we can write

$$F(r) = \int_0^1 \sin r\lambda S(\lambda)\,d\lambda, \tag{3.8}$$

where

$$S(\lambda)=4i\int_{\lambda}^{1} Q(\rho)\,\frac{\lambda}{(\rho^2-\lambda^2)^{1/2}}\,d\rho. \tag{3.9}$$

From this expression of F as a Fourier sine transform we see that F is an odd function of exponential type $\leqslant 1$ with $F(0)=0$. From (3.5) then we have that G is an even function of exponential type $\leqslant 1$ with $G(0)=1$. Such a G can, in view of (3.4), be expressed as a Fourier cosine transform

$$G(r)=\int_{-\infty}^{\infty} \cos rtg(t)\,dt$$

of a continuous even function g whose support is contained in $[-1, 1]$ and which satisfies $\int_{-1}^{1} g(t)\,dt=1$. These properties of G mean that we can also write

$$G(r)=2\int_{0}^{1} \cos rtg(t)\,dt. \tag{3.10}$$

To summarize the analysis thus far: we have associated with a function p satisfying (P1)–(P3) a function G via (3.3) and (3.5); this G has the properties enumerated in the preceding paragraph.

We will now show that every G with these properties (i.e., every even function G in $L_1(\mathbb{R})$ of exponential type $\leqslant 1$ with $G(0)=1$) can be obtained in this way. Assume then that G is any such function. Represent it as (3.10), where g is a constant multiple of $\hat{G}$, here continuous because G is in L_1. Define f on the s-plane ($s=re^{i\theta}$) by

$$f(s)=\frac{-1}{2\pi i}\,\frac{G(r)}{s}=e^{-i\theta}\,\frac{-1}{2\pi i}\,\frac{G(r)}{r}. \tag{3.11}$$

Then $f\in L_1(\mathbb{R}^2)$, indeed $\|f\|_1=\int_0^\infty |G(r)|\,dr$. Now define F from G via (3.5). Then put

$$p(s)=\frac{1}{2\pi i}\,e^{-i\theta}F(r). \tag{3.12}$$

Note that F can be recovered from p via (3.3). The equation (3.11) can be rewritten as

$$f(s)=\frac{-1}{2\pi is}+p(s). \tag{3.13}$$

If we define $S(\lambda)=2\int_\lambda^1 g(t)\,dt$, then using (3.10) and integrating by parts we get $F(r)=\int_0^1 \sin r\lambda S(\lambda)\,d\lambda$. It remains to show that $\operatorname{Supp}\hat{p}\subset D$.

Once again by standard results in Fourier analysis [12, p. 137] any f in $L_1(\mathbb{R}^2)$ which has special angular dependence as in (3.11) has a special kind of Fourier transform. The convenient form of the transform for our purpose is

$$\begin{aligned}\hat{f}(\xi) &= e^{-i\phi} \int_0^\infty G(r)\, J_1(\rho r)\, dr \\ &= e^{-i\phi} \int_0^\infty (1 - rF(r))\, J_1(\rho r)\, dr, \end{aligned} \tag{3.14}$$

where J_1 is the Bessel function of order 1, and as before $\xi = \rho e^{i\phi}$. We want to show that for $|\xi| \geqslant 1$ we have $0 = \hat{p}(\xi) = \hat{f}(\xi) - 1/\xi$. (Here we have used (3.13) and as before taken the Fourier transform in the space $\mathscr{S}'$.) By (3.14) this reduces to proving

$$\int_0^\infty G(r)\, J_1(\rho r)\, dr = \frac{1}{\rho} \tag{3.15}$$

for each $\rho > 1$. Since $J_1 = -J_0'$, we have

$$\int_0^M G(r)\, J_1(\rho r)\, dr = \left[\frac{-J_0(\rho r)}{\rho}\, G(r)\right]_0^M + \frac{1}{\rho} \int_0^M J_0(\rho r)\, G'(r)\, dr.$$

As $M \to \infty$ the first term on the right hand side goes to $1/\rho$. So our claim will be proved if we show

$$\lim_{M \to \infty} \int_0^M J_0(\rho r)\, G'(r)\, dr = 0.$$

Using the representation (3.10) this reduces to proving that for every $\rho > 1$ we have

$$\lim_{M \to \infty} \int_0^1 tg(t)\, dr \int_0^M J_0(\rho r) \sin rt\, dr = 0.$$

By a known fact in Fourier analysis ([14], see (7.4.3) and the subsequent discussion) we can conclude that for $\rho > t$

$$\lim_{M \to \infty} \int_0^M J_0(\rho r) \sin rt\, dr = 0.$$

So our claim will be proved if we show that the integrals

$$I_M(t) = \int_0^M J_0(\rho r) \sin rt\, dr$$

are uniformly bounded for $0 \leqslant t \leqslant 1$ as $M \to \infty$, for each fixed $\rho > 1$. Since the integrals $\int_0^1 J_0(\rho r) \sin rt \, dr$ are surely uniformly bounded we need only show that the integrals $\int_1^M J_0(\rho r) \sin rt \, dr$ are uniformly bounded. For this we use the representation [6, p. 69]

$$J_0(\rho r) = \frac{2}{\pi} \int_1^\infty \frac{\sin \rho r y}{(y^2-1)^{1/2}} \, dy.$$

It is easy to see that for a fixed M the integrals $\int_1^A (\sin \rho r y/(y^2-1)^{1/2}) \, dy$ are uniformly bounded for $1 \leqslant r \leqslant M$ as $A \to \infty$. Indeed, we have for each $r \geqslant 1$ and for $1 < A < B$, by the second mean-value theorem,

$$\left| \int_A^B \frac{\sin \rho r y}{(y^2-1)^{1/2}} \, dy \right| \leqslant \frac{3}{\rho (A^2-1)^{1/2}}.$$

Hence, by the bounded convergence theorem we have

$$\begin{aligned} &\int_1^M J_0(\rho r) \sin rt \, dr \\ &\quad = \frac{2}{\pi} \lim_{A \to \infty} \int_1^M \int_1^A \frac{\sin \rho r y \sin rt}{(y^2-1)^{1/2}} \, dy \, dr \\ &\quad = \frac{1}{\pi} \lim_{A \to \infty} \int_1^A \int_1^M \frac{\cos(\rho y - t) r - \cos(\rho y + t) r}{(y^2-1)^{1/2}} \, dr \, dy. \end{aligned}$$

Doing the inner integration, then estimating the integrand in the remaining integral, using the restrictions $0 \leqslant t \leqslant 1$ and $\rho > 1$ we obtain from this

$$\begin{aligned} \left| \int_1^M J_0(\rho r) \sin rt \, dr \right| &\leqslant \frac{4}{\pi} \frac{1}{\rho - t} \int_1^\infty \frac{dy}{y(y^2-1)^{1/2}} \\ &= \frac{2}{\rho - t} \end{aligned}$$

for $0 \leqslant t \leqslant 1$.

This proves that $I_M(t)$ are uniformly bounded as desired. Hence we have (3.15) and our claim about the support of $\hat{\rho}$ is established.

The conclusion of the above analysis is the following

THEOREM. *Let c_2 be the constant defined by* (1.2). *Then c_2 is also given by*

$$c_2 = \inf \int_0^\infty |G(r)| \, dr, \tag{3.16}$$

where G varies over all L_1 functions of the form $G(r) = 2\int_0^1 \cos rt\, g(t)\, dt$, where g is a continuous even function supported in $[-1, 1]$ such that $\int_{-1}^{1} g(t)\, dt = 1$.

4. The Value of the Two-Variable Constant

The extremal problem we are led to by the considerations of Section 3 is of the general sort discussed in Section 2. We have not so far been able to find c_2 exactly, in spite of the availability of a general machinery for attacking such problems. The background in matrix theory shows that $c_2 \geqq \pi/2$ and suggests that it is larger, see Section 1.

Any function g satisfying the conditions of Section 3 will give us an upper bound on c_2. We present in some detail this especially clear-cut example:

$$g(t) = \frac{\pi}{4} \cos \frac{\pi}{2} t \qquad (|t| \leqq 1),$$

$$G(r) = \pi^2 \frac{\cos r}{\pi^2 - 4r^2}.$$

We want $\int_0^\infty |G(r)|\, dr$. We have

$$\int_0^\infty |G(r)|\, dr = \pi^2 (I_1 + I_2),$$

$$I_1 = \int_0^{\pi/2} \frac{\cos r}{\pi^2 - 4r^2}\, dr,$$

$$I_2 = \int_{\pi/2}^\infty \frac{|\cos r|}{4r^2 - \pi^2}\, dr.$$

These can be simplified by changes of variable.

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_0^{\pi/2} \left(\frac{\cos r}{\pi - 2r} + \frac{\cos r}{\pi + 2r} \right) dr \\ &= \frac{1}{2\pi} \left\{ \int_0^{\pi/2} \frac{\sin u}{2u}\, du + \int_{\pi/2}^{\pi} \frac{\sin v}{2v}\, dv \right\} = \frac{1}{4\pi} \operatorname{Si}(\pi), \\ I_2 &= \frac{1}{2\pi} \int_{\pi/2}^\infty \left(\frac{|\cos r|}{2r - \pi} - \frac{|\cos r|}{2r + \pi} \right) dr \\ &= \frac{1}{4\pi} \left\{ \int_0^\infty \frac{|\sin u|}{u}\, du - \int_\pi^\infty \frac{|\sin v|}{v}\, dv \right\} = \frac{1}{4\pi} \operatorname{Si}(\pi), \end{aligned}$$

where $\operatorname{Si}(x) = \int_0^x (\sin t/t)\, dt$.

Combining,

$$\int_0^\infty |G(r)|\,dr = \pi^2(I_1 + I_2) = \frac{\pi}{2}\,\mathrm{Si}(\pi) < 2.90901$$

(see, e.g., [1]), which is therefore an upper bound for c_2.

Some other candidates which gave numerical values extremely close to this are

$$g(t) = \frac{4}{3}(1 - t^2),\; g(t) = \frac{27}{28}\frac{\pi}{4}\left(\cos\frac{\pi}{2}t - \frac{1}{9}\cos\frac{3\pi}{2}t\right).$$

The only bound we get from below is from general principles, as follows.

Suppose if possible that $\int_0^\infty |G(r)|\,dr \leqslant \pi/2$. This would require the inverse Fourier transform g to satisfy $|g(t)| \leqq \frac{1}{2}$ everywhere. But g is continuous, $g(-1) = g(1) = 0$, and $\int_{-1}^{1} g(t)\,dt = 1$, so this cannot be. Furthermore, the infimum in (3.16) is attained, because the candidate functions G form a normal family (see, e.g., [9, p. 300]). Conclusion: $c_2 > \pi/2$.

To sum up, we have shown in this section that

$$\frac{\pi}{2} < c_2 \leqslant \frac{\pi}{2}\,\mathrm{Si}(\pi).$$

Acknowledgments

The authors express their thanks to Alan McIntosh for his role in initiating this investigation, and to P. G. Rooney and M. G. Nadkarni for illuminating conversations.

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